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Spin^c structures on Hantzsche-Wendt manifolds

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ABSTRACT

Using a combinatorial description of Stiefel-Whitney classes of closed flat manifolds with diagonal holonomy representation, we show that no Hantzsche-Wendt manifold of dimension greater than three admits a spin^c structure.

1. Introduction

Hantzsche-Wendt manifolds are examples of flat manifolds, i.e. closed Riemannian manifolds with vanishing sectional curvature. They are generalizations of the three-dimensional flat orientable manifold defined in [5] and, following [16], we say that:

An orientable n-dimensional flat manifold is Hantzsche-Wendt if and only if its holonomy group is an elementary abelian 2-group of rank $n-1$.

Every n-dimensional flat manifold X occurs as a quotient space of the action of Γ on the euclidean space $\mathbb{R}^n$, where Γ is a Bieberbach group, i.e. a torsion-free, co-compact and discrete subgroup of the group $\mathrm{Isom}(\mathbb{R}^n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ of isometries of $\mathbb{R}^n$. X is an Eilenberg-MacLane space of type $K(\Gamma, 1)$. By Bieberbach theorems (see [19]), Γ is defined by the following short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \stackrel{\iota}{\longrightarrow} \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1, \tag{1.1}$$

where $\iota(\mathbb{Z}^n)$ is the maximal abelian normal subgroup of Γ, G is finite and coincides with the holonomy group of X. Moreover, by conjugations in Γ, G acts in a natural way on $\mathbb{Z}^n$, giving it the structure of a G-module.

Taking into account the above definition we will say that a Bieberbach group $\Gamma \subset \mathrm{Isom}^+(\mathbb{R}^n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n$, defined by (1.1), is a *Hantzsche-Wendt group* and $X = \mathbb{R}^n/\Gamma$ is a *Hantzsche-Wendt manifold* (*HW-group* and *HW-manifold* for short) if $G \simeq C_2^{n-1}$.

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Among many properties of HW-manifolds which were objects of research one can list the following: they exist only in odd dimensions [12], they are rational homology spheres [18] and cohomologically rigid [13]. If Γ is an HW-group then it is an epimorphic image of a certain Fibonacci group [8] and if its dimension is greater than or equal to 5, then its commutator and translation subgroups coincide [14]. One of the crucial – for the purposes of this paper – properties of HW-groups is the one described in [16]: they are *diagonal*, i.e. there exists a $\mathbb{Z}$-basis $\mathcal{B}$ of the G-module $\mathbb{Z}^n$ such that

$$gb = \pm b$$

for every $b \in \mathcal{B}$ and $g \in G$.

Now, let $n \geq 3$. The fundamental group $\pi_1(\mathrm{SO}(n))$ of the special orthogonal group $\mathrm{SO}(n)$ is of order 2. The spin group $\mathrm{Spin}(n)$ is its double cover – and the universal cover in fact. Let $\lambda_n\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be the covering map. A spin structure on a smooth orientable manifold X is an equivariant lift of its frame bundle via λ_n. Its existence is equivalent to the vanishing of the second Stiefel-Whitney class $w_2(X)$ of X, see [3, page 40]. In the case when X is flat, it is closely connected to the Sylow 2-subgroup of its holonomy group [2] and can be determined by an algorithm [9]. The three-dimensional HW-manifold has a spin structure (see [7, Theorem VII.1]). But this is the only case – by [11, Example 4.6] no other HW-manifold admits any spin structure.

One can consider the complex analogue of spin structures. We have that

$$\mathrm{Spin}^c(n) := \left(\mathrm{Spin}(n) \times S^1\right) / \langle(-1,-1)\rangle = \mathrm{Spin}(n) \times_{C_2} S^1$$

is the double cover of $\mathrm{SO}(n) \times S^1$ for which the spin^c structure is defined – in analogy to the spin case – with the covering map $\bar{\lambda}_n\colon \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times S^1$ given by

$$\bar{\lambda}_n[x,z] := \left(\lambda_n(x), z^2\right).$$

The manifold X has a spin^c structure if and only if $w_2(X)$ is the mod 2 reduction of some integral cohomology class $z \in H^2(X,\mathbb{Z})$, see [3, page 49]. We immediately get that existence of spin structures determines existence of spin^c structures – in fact the former induces the latter, but not the other way around. For example, by an unpublished work [20] all orientable 4-manifolds have some spin^c structures, but by [15], 3 of the 27 flat ones don't have any spin structure.

In this paper we prove that every HW-manifold of dimension greater than or equal to 5 does not admit any spin^c structure. Note that some examples of non-spin^c HW-manifolds were given in [4].

The tools that we use have been introduced in [13] and used for example in [10]. They proved their effectiveness in cohomology-related properties of diagonal manifolds.

The structure of the paper is as follows. Sections 2 and 3 give a quick glance on a way of the encoding diagonal manifolds and their Stiefel-Whitney classes by certain matrices. This has been already presented in more detail in [13] and [10]. In Section 4 we give one of two theorems on conditions equivalent to the existence of spin^c structures on HW-manifolds. For our further analysis we introduce HW-matrices. This description of HW-manifolds was introduced in [13] and is in fact one-to-one with the one given in [12]. Technical Section 6 gives us some properties and formulas for matrices that we work with. The second theorem on conditions equivalent to the existence of spin^c structures on HW-manifolds is given in Section 7. After that we give a very specific form to a matrix which describes a (possible) spin^c HW-manifold and at last we show that this form can never occur. This proves that no HW-manifold can admit a spin^c structure.

2. Diagonal Bieberbach groups

In this section we give a combinatorial description of diagonal flat manifolds. This language is essential in the analysis of the Stiefel-Whitney classes of such manifolds.

Remark 2.1. For any matrix A by $A_{ij}, A_{i,j}$ or $A_i[j]$ we shall denote the element in the i-th row and j-th column of A. By A_i we shall understand the i-th row of A.

Remark 2.2. Let $k \in \mathbb{N}$. Cyclic groups of order k with multiplicative and additive structure will be denoted by C_k and $\mathbb{Z}_k := \mathbb{Z}/k$, respectively. Note that in the natural way $\mathbb{Z}_k$ is ring and possibly – a field.

Suppose Γ is a Bieberbach group defined by the short exact sequence (1.1). As mentioned in the introduction, conjugations in Γ define a G-module $\mathbb{Z}^n$. To be a bit more precise, corresponding representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{Z})$ is called an *integral holonomy representation of* Γ and it is given by the formula

$$\rho_g(z) = \iota^{-1}(\gamma\iota(z)\gamma^{-1}),$$

where $z \in \mathbb{Z}^n, g \in G$ and $\gamma \in \Gamma$ is such that $\pi(\gamma) = g$. We will call Γ *diagonal* or *of diagonal type* if the image of ρ is a subgroup of the group

$$D = \{A \in \mathrm{GL}(n, \mathbb{Z}) : A_{ij} = A_{ji} = 0 \text{ and } A_{ii} = \pm 1 \text{ for } 1 \le i < j \le n\} \cong C_2^n$$

of diagonal matrices of $\mathrm{GL}(n,\mathbb{Z})$. Since Γ is torsion-free, $-I \notin \rho(G)$, where I is the identity matrix (see [19, page 133]). It follows that $G = C_2^k$ for some $1 \le k \le n-1$.

Let $S^1 = \mathbb{R}/\mathbb{Z}$. As in [13] and [10], we consider the automorphisms $g_i : S^1 \to S^1$, given by

$$g_0([t]) = [t],\;\; g_1([t]) = \left[t + \frac{1}{2}\right],\;\; g_2([t]) = [-t],\;\; g_3([t]) = \left[-t + \frac{1}{2}\right], \tag{2.1}$$

for $t \in \mathbb{R}$. Let $\mathcal{D} = \{g_i \mid i = 0,1,2,3\}$. It is easy to see that $\mathcal{D} \cong C_2 \times C_2$ and $g_3 = g_1 g_2$. We define an action of $\mathcal{D}^n$ on the torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$ by

$$(t_1, \dots, t_n)(z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n), \tag{2.2}$$

for $(t_1,\dots,t_n) \in \mathcal{D}^n$ and $(z_1,\dots,z_n) \in T^n$.

Any minimal set of generators of a group $C_2^d \subseteq \mathcal{D}^n$ defines a $(d \times n)$-matrix with entries in $\mathcal{D}$ which in turn defines a matrix A with entries in the set $\mathcal{V} = \{\mathit{0}, \mathit{1}, \mathit{2}, \mathit{3}\}$ under the identification $i \leftrightarrow g_i$, $0 \le i \le 3$. Note that elements of $\mathcal{V}$ are written in italic.

Definition 2.3. The structure of an additive group on $\mathcal{V}$ is given by

$$i + j = k \Leftrightarrow g_i g_j = g_k,$$

for $i, j, k \in \mathcal{V}$. This way $\mathcal{V} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is in the natural way a $\mathbb{Z}_2$-vector space.

Example 2.4. The three-dimensional HW-group has generators:

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}\right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right),$$

hence the corresponding matrix $A \in \mathcal{V}^{2\times 3}$ is of the form

$$A = \begin{bmatrix} \mathit{1} & \mathit{3} & \mathit{2} \\ \mathit{2} & \mathit{1} & \mathit{3} \end{bmatrix}.$$

Remark 2.5. Whenever our calculations involve $\mathbb{Z}_2 = \{0, 1\}$ and $\mathcal{V}$, it is done by identifying $\mathbb{Z}_2$ with the subgroup $\{\mathit{0}, \mathit{1}\} < \mathcal{V}$.

We have the following characterization of the action of C_2^d on T^n and the associated orbit space T^n/C_2^d via the matrix A.

Lemma 2.6 *([13, page 1050]). Let $C_2^d \subseteq \mathcal{D}^n$ and define the matrix $A \in \mathcal{V}^{d\times n}$ as above. Then:*

(i) the action of C_2^d on T^n is free if and only if there is $\mathit{1}$ in the sum of any distinct collection of rows of A,
(ii) C_2^d is the holonomy group of T^n/C_2^d if and only if there is either $\mathit{2}$ or $\mathit{3}$ in the sum of any distinct collection of rows of A.

When the action of C_2^d on T^n defined by (2.2) is free, we will say that the associated matrix A is *free* and we will call it the *defining matrix* of T^n/C_2^d. In addition, when C_2^d is the holonomy group of T^n/C_2^d, we will say that A is *effective*.

Corollary 2.7. *Up to affine equivalence, every flat manifold with diagonal fundamental group can be encoded by a defining and effective matrix.*

Proof. Let $\Gamma \subset \mathrm{Isom}(\mathbb{R}^n)$ be an n-dimensional diagonal Bieberbach group defined by the short exact sequence (1.1), where $G = C_2^d$ for some $d < n$. Up to isomorphism, Γ may be in such a form, that it is diagonal and the monomorphism $\iota\colon \mathbb{Z}^n \to \Gamma$ is given by the formula

$$z \mapsto \begin{bmatrix} I & z \\ 0 & 1 \end{bmatrix}.$$

Hence if $(A, a) \in \Gamma$, then $A \in D$ and $a \in \frac{1}{2}\mathbb{Z}^n$. Now, every flat manifold with the fundamental group isomorphic to Γ is affine equivalent to

$$X = \mathbb{R}^n/\Gamma = (\mathbb{R}^n/\mathbb{Z}^n)/(\Gamma/\mathbb{Z}^n) = T^n/G.$$

By the above description of elements of Γ, we get $G \subset \mathcal{D}^n$. Let $\in \mathcal{V}^{d\times n}$ be a matrix defined by G. Since G is a holonomy group of a flat manifold X, A is defining and effective. □

3. Stiefel-Whitney classes of diagonal flat manifolds

The goal of this section is to introduce a notation and some basic results on Stiefel-Whitney classes of diagonal flat manifolds. For more precise description see [10] and [13].

Let $n \in \mathbb{N}$ and Γ be an n-dimensional diagonal Bieberbach group, given by the extension (1.1), with non-trivial holonomy group $G = C_2^d$ $(d > 0)$. Let $A \in \mathcal{V}^{d\times n}$ be a defining matrix of the corresponding flat manifold $X = \mathbb{R}^n/\Gamma = T^n/C_2^d$.

It is well-known that

$$H^*(C_2^d; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \dots, x_d],$$

where $\{x_1, \dots, x_d\}$ is a basis of $H^1(C_2^d, \mathbb{Z}_2) = \mathrm{Hom}(C_2^d, \mathbb{Z}_2)$ (see [1, Theorem 1.2]). Let

$$\pi^* \colon H^*(C_2^d, \mathbb{Z}_2) \to H^*(\Gamma, \mathbb{Z}_2)$$

be the induced cohomology ring homomorphism. By [10, Proposition 3.2] the total Stiefel-Whitney class is given by

$$w(X) = \pi^*(\mathrm{sw}) \in H^*(\Gamma, \mathbb{Z}_2),$$

where

$$\mathrm{sw} = \prod_{j=1}^{n} (1 + \alpha_j + \beta_j). \tag{3.1}$$

In the above formula for every $1 \le j \le n$, $\alpha_j, \beta_j \in H^1(C_2^d, \mathbb{Z}_2)$ are the cocycles defined by

$$\alpha_j = \sum_{k=1}^{d} \alpha(A_{kj})x_k, \ \beta_j = \sum_{k=1}^{d} \beta(A_{kj})x_k,$$

where the linear homomorphisms $\alpha, \beta \in \mathrm{Hom}_{\mathbb{Z}_2}(\mathcal{V}, \mathbb{Z}_2)$ are uniquely defined by the following rules

$$\alpha(2) = \beta(3) = 1 \text{ and } \alpha(3) = \beta(2) = 0. \tag{3.2}$$

Let

$$\pi^*_{(i)} \colon H^i(C_2^{n-1}, \mathbb{Z}_2) \to H^i(\Gamma, \mathbb{Z}_2)$$

be the induced group cohomology homomorphism (restriction of π^* to the i-th gradation), for $0 \le i \le n$. Using again [10, Proposition 3.2] and the five-term exact sequence for the extension (1.1) (see [10, Formula (7)]) we get

Lemma 3.1. *$\pi^*_{(1)}$ is injective and the kernel of $\pi^*_{(2)}$ is spanned by*

$$\theta_j = \alpha_j \cup \beta_j = \alpha_j\beta_j$$

for $1 \le j \le n$.

Remark 3.2. Note that the polynomials $\mathrm{sw}, \alpha_j, \beta_j, \theta_j$, where $1 \le j \le n$, can be defined for any matrix $A \in \mathcal{V}^{d\times n}$. To emphasize this connection or in the case when it won't be clear from the context, we will add the superscript A to them and write sw^A for example.

4. Bockstein maps and spin^c structures

We will keep the notation of the previous section and restrict our attention to the case of Hantzsche-Wendt manifolds of dimension greater than or equal to 5. Hence $n \ge 5$ is an odd integer and $d = n - 1$. Let β_Γ and $\tilde{\beta}_\Gamma$ be the Bockstein homomorphisms of cohomology groups of Γ associated to the short exact sequences

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\mathrm{mod}_2} \mathbb{Z}_2 \longrightarrow 0 \tag{4.1}$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\mathrm{mod}_2} \mathbb{Z}_2 \longrightarrow 0 \tag{4.2}$$

respectively. If $\rho \colon H^2(\Gamma, \mathbb{Z}) \to H^2(\Gamma, \mathbb{Z}_2)$ is the homomorphism induced by the mod 2 map, then we have the following commutative diagram

$$\begin{array}{ccccc} H^1(\Gamma,\mathbb{Z}) & \longrightarrow & H^1(\Gamma,\mathbb{Z}_2) & \xrightarrow{\tilde\beta_\Gamma} & H^2(\Gamma,\mathbb{Z}) \\ & & & \searrow^{\beta_\Gamma} & \downarrow \rho \\ & & & & H^2(\Gamma,\mathbb{Z}_2) \end{array}$$

with the row forming an exact sequence (see [6, Chapter 3.E]). By [14, Theorem 3.1] $H_1(\Gamma) \cong \mathbb{Z}_2^{n-1}$. By [19, Theorem 9.2] $H_2(\Gamma)$ is a finite group. Moreover from the universal coefficient theorem ([6, Theorem 3.2]),

$$H^1(\Gamma,\mathbb{Z}) = 0 \text{ and } H^1(\Gamma,\mathbb{Z}_2) \cong H^2(\Gamma,\mathbb{Z}) \cong \mathbb{Z}_2^{n-1}.$$

Hence $\tilde\beta_\Gamma$ is an isomorphism and $\operatorname{Im}\beta_\Gamma = \operatorname{Im}\rho$.

Let β be the Bockstein homomorphism of cohomology groups of C_2^{n-1} associated to the extension (4.1). The homomorphism π induces the commutative diagram

$$\begin{array}{ccc} H^1(C_2^{n-1},\mathbb{Z}_2) & \xrightarrow{\beta} & H^2(C_2^{n-1},\mathbb{Z}_2) \\ \downarrow \pi^*_{(1)} & & \downarrow \pi^*_{(2)} \\ H^1(\Gamma,\mathbb{Z}_2) & \xrightarrow{\beta_\Gamma} & H^2(\Gamma,\mathbb{Z}_2) \end{array}$$

By Lemma 3.1, $\pi^*_{(1)}$ is a monomorphism of the elementary abelian 2-groups of rank $n-1$, hence it is an isomorphism and

$$\operatorname{Im}\rho = \operatorname{Im}\beta_\Gamma = \operatorname{Im}\beta_\Gamma\pi^*_{(1)} = \operatorname{Im}\pi^*_{(2)}\beta = \operatorname{Im}\pi^*\beta.$$

Let sw_2 be the sum of degree 2 terms of the polynomial sw. Then $w_2(X) = \pi^*(\mathrm{sw}_2)$ and by the above calculations the manifold $X = \mathbb{R}^n/\Gamma$ admits a spin^c structure if and only if $\pi^*(\mathrm{sw}_2) \in \operatorname{Im}\pi^*\beta$. This condition is obviously equivalent to

$$(\mathrm{sw}_2 + \ker\pi^*) \cap \operatorname{Im}\beta \neq \emptyset.$$

In addition, one can easily show that for every $x \in H^1(C_2^{n-1},\mathbb{Z}_2)$ and $a, b \in C_2^{n-1}$ we have

$$\beta(x)(a,b) = x(a)x(b) = x^2(a,b),$$

hence $\beta(x) = x^2$ and $\pi^*(\beta(x)) = \pi^*(x)^2$. Similarly, $\beta_\Gamma(f) = f^2$ for $f \in H^1(\Gamma,\mathbb{Z}_2)$.

With usage of Lemma 3.1, we collect results of this section in the following theorem:

Theorem 4.1. *Assume that $n \geq 5$ is an odd integer and X is an n-dimensional Hantzsche-Wendt manifold. Let $A \in \mathcal{V}^{n-1\times n}$ be a defining matrix of X. Then the following conditions are equivalent:*

1. *X admits a spin^c structure.*
2. *$w_2(X) \in H^*(\Gamma,\mathbb{Z}_2)$ is a square.*
3. *There exists $x \in H^1(\mathbb{Z}_2^{n-1},\mathbb{Z}_2)$ such that $x^2 + \mathrm{sw}_2^A \in \operatorname{span}\{\theta_1^A,\dots,\theta_n^A\}$.*

5. HW matrices

Let $n \in \mathbb{N}$. Every n-dimensional HW-manifold X defines some matrix $A \in \mathcal{V}^{n-1\times n}$. For the purpose of investigating spin^c properties of X it will be more convenient to work with a square matrix – an HW-matrix. HW-matrices were defined in [13].

Let Z be a finite set. By $\mathcal{P}(Z)$ we denote the Boolean algebra of subsets of Z. Just recall that the addition and multiplication in $\mathcal{P}(Z)$ are defined by the symmetric difference and intersection respectively:

$$\forall_{A,B\in\mathcal{P}(Z)} A + B := (A \setminus B) \cup (B \setminus A) \text{ and } A \cdot B := A \cap B.$$

Empty set and Z are zero and one of this algebra, respectively. Let us note without a proof:

Lemma 5.1. *Let Z be a finite set.*

1. *The map $|\cdot|_2\colon \mathcal{P}(Z) \to \mathbb{Z}_2$, given by*

   $$U \mapsto |U| \bmod 2,$$

   is linear.
2. *Every permutation of Z is an algebra automorphism of $\mathcal{P}(Z)$.*

Remark 5.2. We will use the notation $\mathcal{P}_d := \mathcal{P}(\{1,\dots,d\})$ for $d \in \mathbb{N}$.

Definition 5.3. Let $d, n \in \mathbb{N}$ and $A \in \mathcal{V}^{d\times n}$. For $S \in \mathcal{P}_n$ and $1 \le i \le d$ we have the sum of elements of the i-th row A which lie in the columns from the set S:

$$\mathrm{smr}_i^S(A) := \sum_{j\in S} A_{ij}$$

and we denote $\mathrm{smr}_i^{\{1,\dots,n\}}(A)$ simply by $\mathrm{smr}_i(A)$. In a similar way we define the column sums $\mathrm{smc}_j^S(A)$ (and $\mathrm{smc}_j(A)$) for $S \in \mathcal{P}_d$ and $1 \le j \le n$. Moreover, we define a map $J_A \colon \mathcal{P}_d \to \mathcal{P}_n$ as follows

$$J_A(U) := \left\{ j : \mathrm{smc}_j^U(A) = 1 \right\}.$$

Definition 5.4. There exists the unique $\mathbb{Z}_2$-linear involution $\bar{\cdot} \colon \mathcal{V} \to \mathcal{V}$ which maps 2 to 3. We call this map a *conjugation*. To be explicit, we have

$$\overline{0} = 0, \overline{1} = 1, \overline{2} = 3 \text{ and } \overline{3} = 2.$$

Definition 5.5. Let $d, n \in \mathbb{N}$ and $A \in \mathcal{V}^{d\times n}$. We call A:

- *self-conjugate* if $A^t = \overline{A}$, where A^t is the transpose of A and $\overline{A}$ is the element-wise conjugate of A;
- *distinguished* if

  $$A_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 2 \text{ or } 3 & \text{if } i \neq j, \end{cases}$$

  for $1 \le i \le d, 1 \le j \le n$.

Remark 5.6. Recall that we speak about a *principal submatrix* of a given matrix if the sets of row and column indices which define it are the same (see [17, Definition 6.2.5] for example). We immediately get, that principal submatrices of self-conjugate and distinguished matrices are themselves self-conjugate and distinguished, respectively.

Lemma 5.7. *Let $A \in \mathcal{V}^{k\times n}$ be distinguished, where $k \le n$. Then the possible values for $\mathrm{smc}_j(A)$, where $1 \le j \le n$ are given by the following table:*

	$j \le k$	$j > k$
$2 \mid k$	2 or 3	0 or 1
$2 \nmid k$	0 or 1	2 or 3

Proof. Let $a, b \in \mathbb{N}$. The sum $a \cdot 2 + b \cdot 3$ depends only on the parity of a and b in the following way:

b \ a	even	odd
even	0	2
odd	3	1

In order to prove the lemma we use the above table. For $j \le k$ we take $a + b = k - 1$ and consider possible values for the sum $1 + a \cdot 2 + b \cdot 3$, while for $j > k$ we take $a + b = k$ and the sum $a \cdot 2 + b \cdot 3$. □

Definition 5.8 *([13, Definition 2]).* Let $n \in \mathbb{N}$. We will call $A \in \mathcal{V}^{n\times n}$ a *HW-matrix* if:

1) A is distinguished;
2) $\mathrm{smc}_j(A) = 0$ for every $1 \le j \le n$;
3) $J_A(U) \neq 0$ for every $U \in \mathcal{P}_n \setminus \{0, 1\}$.

The set of HW-matrices of degree n, or n-HW-matrices for short, will be denoted by $\mathcal{H}_n$.

By Lemma 5.7 we immediately get:

Corollary 5.9. *Every HW-matrix is of odd degree.*

Directly from the definition of HW-matrices, we get the following properties of the map J:

Corollary 5.10 *([13, Proposition 3]). Let M be an HW-matrix of degree n. Then:*

1) $J_M(1) = 0$;
2) $J_M(U) = J_M(1 + U)$.

Proof. First property is just a reformulation of condition 2) of Definition 5.8. The second property follows from the fact that for every $1 \le j \le n$ we have

$$\mathrm{smc}_j^U(M) + \mathrm{smc}_j^{1+U} = \mathrm{smc}_j(M) = 0. \quad \square$$

Remark 5.11. We can think of Definition 5.8 as coming from the encoding of Hantzsche-Wendt groups presented in [12]. In connection to this description we note:

1. Any row of an HW-matrix may be removed and the corresponding torus quotient will remain the same. In other words, the removal will make the matrix a defining and effective one for the same HW-manifold.
2. Every HW-manifold defines some HW-matrix (see Corollary 2.7).
3. There is an action of the group $G_n := C_2 \wr S_n = C_2^n \rtimes S_n$ on the set $\mathcal{V}^{n\times n}$. Namely, for every $A \in \mathcal{V}^{n\times n}$ we have that
   (a) c_k conjugates the k-th column of A, where $c_k \in C_2^n$ has non-trivial element of C_2 in the k-th coordinate only;
   (b) $\sigma \cdot A := P_\sigma A P_\sigma^{-1}$, where $P_\sigma \in \mathrm{GL}_n(\mathbb{Z})$ is the permutation matrix of $\sigma \in S_n$.

Keeping the above remark in mind, we can reformulate [12, Proposition 1.5] as follows:

Proposition 5.12. *The HW-manifolds X and X', with corresponding HW-matrices $A, A' \in \mathcal{V}^{n\times n}$, are affine equivalent if and only if A and A' are in the same orbit of the action of the group G_n.*

6. Square distinguished matrices

The following section is a bit of a technical nature. Its purpose is to present some properties of square distinguished matrices. We start with a negative result:

Lemma 6.1. *Let $n > 1$ be an integer. There does not exist a matrix $M \in \mathcal{V}^{n\times n}$ such that:*

(A1) M is distinguished and self-conjugate;
(A2) the first row of M is of the form $M_1 = [1, 2, \dots, 2]$;
(A3) $\mathrm{smc}_i M = 1$ for $1 \le i \le n$;
(A4) in every principal submatrix of M of odd degree there exists a column with sum of elements equal to 1.

Proof. Assume that such a matrix M exists. We will list some of its properties.

(P1) Action by permutations of the set $\{2, 3, \dots, n\}$ on M, as in Remark 5.11, does not change its properties (A1)–(A4).
(P2) $\mathrm{smr}_i(M) = 1$ for every $1 \le i \le n$, since

$$\mathrm{smr}_i(M) = \sum_{j=1}^n M_{ij} = \sum_{j=1}^n \overline{M_{ji}} = \overline{\sum_{j=1}^n M_{ji}} = \overline{\mathrm{smc}_i(M)} = \overline{1} = 1.$$

(P3) n is odd, by Lemma 5.7.
(P4) $M_{2,1} = 3$ by self-conjugacy of M.
(P5) The second row of M cannot be of the form $[3, 1, 2, \dots, 2]$, otherwise

$$\mathrm{smr}_2(M) = 3 + 1 + (n-2)2 = 2 + 2 = 0,$$

which contradicts (P2).

(P6) The second row of M cannot be of the form $[3, 1, 3, \dots, 3]$. Otherwise

$$M = \begin{bmatrix} * & A \\ * & B \end{bmatrix}, \text{ where } A = \begin{bmatrix} 2 & \dots & 2 \\ 3 & \dots & 3 \end{bmatrix} \in \mathcal{V}^{2\times n-2}$$

Using (A3), for every $i > 2$ we get

$$1 = \mathrm{smc}_i(M) = 2 + 3 + \mathrm{smc}_{i-2}(B) = 1 + \mathrm{smc}_{i-2}(B),$$

hence $\mathrm{smc}_{i-2}(B) = 0$ and this, together with (P3), contradicts (A4).

(P7) Using (P1), (P5) and (P6), we can assume that

$$M_2 = [3, 1, \underbrace{2, \dots, 2}_{a}, \underbrace{3, \dots, 3}_{b}],$$

where $a, b > 0$. Moreover, a is even (and $b = n - 2 - a$ is odd), since

$$1 = \mathrm{smr}_2(M) = 3 + 1 + a \cdot 2 + b \cdot 3 = 2 + a \cdot 2 + (n - 2 - a) \cdot 3$$
$$= (1 + a) \cdot 2 + (1 + a) \cdot 3 = (1 + a)(2 + 3) = (1 + a) \cdot 1 = 1 + a \cdot 1.$$

(P8) Let M have the following block form[1]

$$M = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & 1 & 2 & 3 \\ * & * & * & C \\ * & * & * & D \end{bmatrix},$$

where on the diagonal we have matrices of degrees $1, 1, a$ and b. There exists an element of C equal to 2. Otherwise, for every $i > a + 2$, we have

$$1 = \mathrm{smc}_i(M) = 2 + 3 + a \cdot 3 + \mathrm{smc}_{i-a-2}(D) = 1 + \mathrm{smc}_{i-a-2}(D)$$

and since D is a principal submatrix of M of odd degree, we get a contradiction with (A4).

By (P8) there exist i and j, such that $3 \le i \le a + 2 < j \le n$ and the principal submatrix Δ of M given by indices $(2, i, j)$ is of the form

$$\Delta = \begin{bmatrix} 1 & 2 & 3 \\ * & 1 & 2 \\ * & * & 1 \end{bmatrix}.$$

By self-conjugacy of Δ we immediately get

$$\Delta = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix},$$

but this contradicts (A4). □

Remark 6.2. To a logical sentence Θ we assign (in a natural way) an element $[\Theta] \in \mathbb{Z}_2$ as follows:

$$[\Theta] = 1 \Leftrightarrow \Theta \text{ is true.}$$

Remark 6.3. Let $n \in \mathbb{N}$, $M \in \mathcal{V}^{n\times n}$ and $U \in \mathcal{P}_n$. By M_U we denote the sum of the rows of M from the set U:

$$M_U := \sum_{i\in U} M_i$$

and $M_U[j]$ – its j-th coordinate, for $1 \le j \le n$. We get

$$J_M(U) = \left\{ j : \mathrm{smc}_j^U(M) = 1 \right\} = \{ j : M_U[j] = 1 \}.$$

[1] Note that in our notation of a block form of a matrix a single element represents a matrix of proper dimension with this element in all its entries.

The following lemma, which describes map J for distinguished matrices, extends [13, Proposition 3]. Recall that we treat $\mathbb{Z}_2$ as a subgroup of $\mathcal{V}$ (see Remark 2.5).

Lemma 6.4. *Let $n \in \mathbb{N}$, $M \in \mathcal{V}^{n\times n}$ be distinguished and $S, U \in \mathcal{P}_n$. The following hold:*

1. $J_M(U) = U$ *if* $|U| = 1$.
2. $J_M(U) \subset U$ *if* $|U|_2 = 1$.
3. $J_M(U) \cdot U = 0$ *if* $|U|_2 = 0$.
4. $|J_M(U)|_2 = \sum_{i,j\in U} M_{ij}$ *if* $|U|_2 = 1$.
5. $|J_M(U)|_2 = \sum_{i,j\in U} M_{ij} + \sum_{i\in U} \mathrm{smr}_i(M)$ *if* $|U|_2 = 0$.
6. $|J_M(U)S|_2 = \sum_{j\in U}[j \in S]M_U[j]$ *if* $|U|_2 = 1$.
7. $|J_M(U)S|_2 = \sum_{j\in U}[j \in S]M_U[j] + \sum_{i\in U} \mathrm{smr}_i^S(M)$ *if* $|U|_2 = 0$.

Proof. Property 1. holds just because M is distinguished – in fact, we have

$$\forall_{1\le i\le n} J_M(\{i\}) = \{i\}. \tag{6.1}$$

Properties 2. and 3. hold by the same rule as in the proof of Lemma 5.7. This rule will be also used in the rest of the proof.

Note that 4. and 5. follow from 6. and 7. respectively, if one takes $S = \{1, \dots, n\} = 1 \in \mathcal{P}_n$.

If $|U|$ is odd then $M_U[j] \in \{0, 1\}$ if and only if $j \in U$ and $[j \in J_M(U)] = M_U[j] \cdot [j \in U]$ for $1 \le j \le n$, hence

$$|J_M(U)S|_2 = \sum_{j=1}^{n}[j \in S][j \in J_M(U)] = \sum_{j=1}^{n}[j \in S][j \in U]M_U[j] = \sum_{j\in U}[j \in S]M_U[j].$$

If $|U|$ is even on the other hand, we get that $M_U[j] \in \{0, 1\}$ if and only if $j \notin U$ and $[j \in J_M(U)] = M_U[j] \cdot [j \notin U]$. In a similar fashion as above we have

$$\begin{aligned}|J_M(U)S|_2 &= \sum_{j=1}^{n}[j \in S][j \in J_M(U)] = \sum_{j=1}^{n}[j \in S][j \notin U]M_U[j]\\ &= \sum_{j\in U}[j \in S]M_U[j] + \sum_{j=1}^{n}[j \in S]M_U[j]\\ &= \sum_{j\in U}[j \in S]M_U[j] + \sum_{j=1}^{n}[j \in S]\sum_{i\in U} M_{ij}\\ &= \sum_{j\in U}[j \in S]M_U[j] + \sum_{i\in U}\sum_{j\in S} M_{ij} = \sum_{j\in U}[j \in S]M_U[j] + \sum_{i\in U} \mathrm{smr}_i^S(M). \quad \square\end{aligned}$$

7. Spin^c structures and HW-matrices

In this section we give a necessary and sufficient condition for the existence of a spin^c structure on a manifold defined by an HW-matrix.

Let us note an easy lemma.

Lemma 7.1. *Let $d \in \mathbb{N}$. A map $\kappa_A : \mathbb{Z}_2[x_1, \dots, x_d] \to \mathrm{Map}(\mathcal{P}_d, \mathbb{Z}_2)$ given by*

$$\kappa_A(x_i)(U) = [i \in U],$$

where $1 \le i \le d$ and $U \in \mathcal{P}_d$, defines an algebra homomorphism. The algebra structure of $\mathrm{Map}(\mathcal{P}_d, \mathbb{Z}_2)$ is given by point-wise addition and multiplication of functions.

We will use the following properties of the map κ_A:

Lemma 7.2. *Let $d, n \in \mathbb{N}$ and $A \in \mathcal{V}^{d\times n}$. Then:*

1) κ_A is a monomorphism in gradation 2;

2) $\kappa_A(\theta_j^A)(U) = [j \in J_A(U)]$.

Proof. Let $\kappa = \kappa_A$ and

$$x = \sum_{1 \le i < j \le d} \alpha_{ij} x_i x_j \in \ker \kappa,$$

where $\alpha_{ij} \in \mathbb{Z}_2$. For any $1 \le k < l \le d$ and $U = \{k, l\}$ we have

$$0 = \kappa \left(\sum_{1 \le i < j \le d} \alpha_{ij} x_i x_j \right)(U) = \sum_{1 \le i < j \le d} \alpha_{ij} \kappa(x_i)(U) \cdot \kappa(x_j)(U) = \alpha_{kl},$$

hence $x = 0$.

Now take $1 \le j \le n$. We have

$$\theta_j = \alpha_j \beta_j = \left(\sum_{i=1}^{d} \alpha(A_{ij}) x_i \right) \left(\sum_{k=1}^{d} \beta(A_{kj}) x_k \right)$$

and in the consequence, for any $U \in \mathcal{P}_d$,

$$\kappa(\theta_j)(U) = \left(\sum_{i \in U} \alpha(A_{ij}) \right) \left(\sum_{k \in U} \beta(A_{kj}) \right).$$

Denote by a, b, c, d the number of $0, 1, 2, 3$ in the rows from the set U of j-th column of A, respectively. We get $\kappa(\theta_j)(U) = (b + c)(b + d) \bmod 2$, but

$$(b + c)(b + d) \bmod 2 = 1 \Leftrightarrow (b + c) \bmod 2 = (b + d) \bmod 2 = 1.$$

Hence $\kappa(\theta_j)(U) = 1$ if and only if

$$\begin{aligned} 1 &= (b + c) \cdot 2 + (b + d) \cdot 3 \\ &= b \cdot (2 + 3) + c \cdot 2 + d \cdot 3 \\ &= a \cdot 0 + b \cdot 1 + c \cdot 2 + d \cdot 3 = \operatorname{smc}_j^U(A), \end{aligned}$$

which by definition means, that $j \in J_A(U)$. □

Proposition 7.3. *Let $n > 1$ be an odd integer and let $A \in \mathcal{V}^{n-1 \times n}$ be distinguished. The following conditions are equivalent:*

1. *There exists $x \in H^1(C_2^{n-1}, \mathbb{Z}_2)$ such that $x^2 + \mathrm{sw}_2^A \in \operatorname{span}\{\theta_1^A, \dots, \theta_n^A\}$.*
2. *$\sigma_2 \in V_\delta := \operatorname{span}\{\theta_1^A - x_1^2, \dots, \theta_{n-1}^A - x_{n-1}^2, \theta_n^A\}$, where σ_2 is the elementary symmetric polynomial of degree 2 in variables $x_1, \dots, x_n$.*
3. *There exists $S \in \mathcal{P}_n$, such that for every $U \in \mathcal{P}_{n-1}$ the equality (in $\mathbb{Z}_2$) holds*

$$|(J_A(U) + U)S|_2 = \binom{|U|}{2}. \tag{7.1}$$

Proof. We will omit the super and subscript A in the proof.

Denote by V the subspace of $\mathbb{Z}_2[x_1, \dots, x_{n-1}]$ of polynomials of degree 2. Let V_s and V_f be subspaces ov V generated by monomials which are and are not squares, respectively. Let $p \colon V \to V_f$ be the projection coming from the decomposition $V = V_s \oplus V_f$. Note that

$$p(\theta_j) = \theta_j - x_j^2 \text{ and } p(\theta_n) = \theta_n$$

for $1 \le j < n$, hence condition 1. is equivalent to

$$p(\mathrm{sw}_2) \in \operatorname{span}\{p(\theta_1^A), \dots, p(\theta_n^A)\} = V_\delta. \tag{7.2}$$

Recall functionals α and β defined by (3.2). Since A is distinguished, we have

$$\alpha(A_{kj}) + \beta(A_{kj}) = \begin{cases} 0 & \text{if } k = j, \\ 1 & \text{if } k \ne j, \end{cases}$$

for $1 \le k \le n-1$ and $1 \le j \le n$. Using formula (3.1) we get

$$\mathrm{sw} = \prod_{j=1}^{n}\left(1+\sum_{k=1}^{n-1}[k \neq j]x_k\right)$$

and in the expansion of this polynomial the expression $x_i x_j$, where $i \neq j$, occurs exactly $(n-2)^2+(n-1)$ times. Since n is odd, we get $p(\mathrm{sw}_2)=\sigma_2$.

Assume $1 \le j \le n$ and let $\delta_j := p(\theta_j)$. For $U \in \mathcal{P}_{n-1}$ we have that

$$\kappa(\delta_j)(U) = [j \in J(U)+U]. \tag{7.3}$$

Indeed, if $j<n$, using Lemma 7.2 we get

$$\begin{aligned}\kappa(\delta_j)(U) &= \kappa(\theta_j + x_j^2)(U) = \kappa(\theta_j)(U)+\kappa(x_j^2)(U)\\ &= \kappa(\theta_j)(U)+\kappa(x_j)(U)^2 = \kappa(\theta_j)(U)+\kappa(x_j)(U)\\ &= [j \in J(U)]+[j\in U] = [j \in J(U)+U].\end{aligned}$$

Additionally, $\delta_n=\theta_n$ and $n \notin U$, hence

$$\kappa(\delta_n)(U) = [n \in J(U)] = [n \in J(U)] + [n \in U] = [n \in J(U)+U].$$

Suppose that $\sigma_2 = \sum s_j\delta_j \in V_\delta$ and let $S := \{j : s_j = 1\} \in \mathcal{P}_n$. For every $U \in \mathcal{P}_{n-1}$ we have

$$\kappa(\sigma_2)(U) = \sum_{j=1}^{n} s_j \kappa(\delta_j)(U).$$

Since

$$\kappa(\sigma_2)(U) = \sum_{1\le k<l<n}[l\in U][k \in U] = \sum_{\substack{k,l\in U\\ k<l}} 1 = \binom{|U|}{2}$$

and

$$\begin{aligned}\sum_{j=1}^{n} s_j\kappa(\delta_j)(U) &= \sum_{j=1}^{n}[j\in S][j \in J(U)+U]\\ &= \sum_{j=1}^{n}[j \in S\cdot(J(U)+U)] = |S\cdot(J(U)+U)|_2,\end{aligned}$$

formula (7.1) follows.

Now assume that (7.1) holds for some $S \in \mathcal{P}_n$ and every $U \in \mathcal{P}_{n-1}$. By the above calculations it may me written as

$$\sum_{j=1}^{n}[j\in S][j\in J(U)+U] = \kappa(\sigma_2)(U).$$

Put $s_j = [j \in S]$ and use (7.3). The above equation takes the form

$$\sum_{j=1}^{n} s_j\kappa(\delta_j)(U) = \kappa(\sigma_2)(U).$$

Recall that U is any element of $\mathcal{P}_{n-1}$. Using this and the linearity of κ, we get

$$\kappa\left(\sum s_j\delta_j\right) = \kappa(\sigma_2).$$

By Lemma 7.2, $\sigma_2 = \sum s_j\delta_j \in V_\delta$. □

Definition 7.4. Let $n\in\mathbb{N}$, $M \in \mathcal{V}^{n\times n}$ and $S \in \mathcal{P}_n$.

1. We call S a *spin^c set* for M if for every $U \in \mathcal{P}_n$ the equation

$$|(J_M(U)+U)S|_2 = \binom{|U|}{2} \tag{7.4}$$

holds.
2. We call S an *almost spin^c set* for M if for every $U \in \mathcal{P}_{n-1}$ equation (7.4) holds.

If S is a spin^c set for M, we call (M, S) a *spin^c pair*.

Lemma 7.5. *Let $n \in \mathbb{N}$ be odd, $M \in \mathcal{H}_n$ and $S \in \mathcal{P}_n$.*

1. If S is an almost spin^c set for M, then

$$|S|_2 = \frac{n-1}{2}.$$

2. If S is an almost spin^c set for M, then it is a spin^c set for M.

Proof. Take $U = \{1, \dots, n-1\}$. By Lemma 6.4 and Corollary 5.10, $J(U) = J(1+U) = J(\{n\}) = \{n\}$. Hence $J(U)+U = \{1, \dots, n\} = 1$, $(J(U)+U)S = S$ and we get

$$|S|_2 = |(J(U)+U)S|_2 = \binom{|U|}{2} = \binom{n-1}{2} = \frac{n-1}{2}.$$

Note again, that all equations above are in $\mathbb{Z}_2$. In particular the last one holds, because n is odd.

Assume now that S is an almost spin^c set for M. Equation (7.4) holds for every $U \in \mathcal{P}_{n-1}$. It is enough to show that it also holds whenever $n \in U$. In that case however $V = 1+U \in \mathcal{P}_{n-1}$, so we have

$$|(J(V)+V)S|_2 = \binom{|V|}{2}.$$

By Corollary 5.10, $J(V) = J(1+U) = J(U)$, hence

$$(J(U)+U)S = (J(V)+V+1)S = (J(V)+V)S + S$$

and by linearity of $|\cdot|_2$ we have

$$\begin{aligned}|(J(U)+U)S|_2 = |(J(V)+V)S|_2 + |S|_2 &= \binom{|V|}{2} + \frac{n-1}{2} \\ &= \binom{n-|U|}{2} + \frac{n-1}{2} = \binom{|U|}{2},\end{aligned}$$

where in the last equality we again use the fact, that n is odd. □

Theorem 7.6. *Let $n \in \mathbb{N}, n \geq 5$, $M \in \mathcal{H}_n$ and let X be the HW-manifold defined by M. The following conditions are equivalent:*

1. X admits a spin^c structure.
2. There exists a spin^c set for M.

Proof. By Lemma 7.5 existence of a spin^c and an almost spin^c set are equivalent conditions. Let A be a matrix composed from the first $n-1$ rows of M. Clearly it is distinguished and by Remark 5.11, A is defining and effective matrix for X. In order to get the desired equivalence, notice that for every $U \in \mathcal{P}_{n-1}$ the equality

$$J_A(U) = J_M(U)$$

holds, use Theorem 4.1 and Proposition 7.3. □

8. Standard forms of spin^c pairs

Recall that in Remark 5.11 we have defined the action of the group $G_n = C_2 \wr S_n$ on the space $\mathcal{V}^{n\times n}$, for every $n \in \mathbb{N}$. We will show that in fact it can act on spin^c pairs.

Lemma 8.1. *Let $n \in \mathbb{N}$, $M \in \mathcal{V}^{n\times n}$, $S \in \mathcal{P}_n$ be such that (M, S) is a spin^c pair. Then for every $\sigma \in S_n$, $(\sigma M, \sigma S)$ is also a spin^c pair.*

Proof. Let $U \in \mathcal{P}_n$ and $\sigma \in S_n$. Using an easy observation that $J_{\sigma M}(U) = \sigma J_M(\sigma^{-1}U)$ and Lemma 5.1, we get

$$\begin{aligned}\left|\big(J_{\sigma M}(U)+U\big)(\sigma S)\right|_2 &= \left|\big(\sigma J_M(\sigma^{-1}U)+U\big)(\sigma S)\right|_2 = \\ &= \left|\sigma\Big(\big(J_M(\sigma^{-1}U)+\sigma^{-1}U\big)S\Big)\right|_2 \\ &= \left|\big(J_M(\sigma^{-1}(U))+\sigma^{-1}(U)\big)S\right|_2 = \binom{|\sigma^{-1}(U)|}{2} = \binom{|U|}{2}. \quad \square\end{aligned}$$

Note, with the assumptions of the above lemma, that G_n acts on $\mathcal{P}_n$ by permutations, using the canonical epimorphism $G_n \to S_n$. Moreover, if $g \in G_n$ is an element which acts by conjugations of columns only, then $J_{gM} = J_M$, since $\overline{1} = 1$. We immediately get

Corollary 8.2. *Let $n \in \mathbb{N}$, $M \in \mathcal{V}^{n\times n}$, $S \in \mathcal{P}_n$ be such that (M, S) is a spin^c pair. Then for every $g \in G_n$, (gM, gS) is also a spin^c pair.*

Lemma 8.3. *Let $n \in \mathbb{N}$ and $M \in \mathcal{V}^{n\times n}$ be distinguished and such that*

$$|J_M(U)|_2 = 1$$

for every two-element set $U \in \mathcal{P}_n$. Then there exists an integer k, such that $2k \geq n$ and in the orbit G_nM there exists a matrix M' in the following block form

$$M' = \begin{bmatrix} A & C \\ C^t & B \end{bmatrix},$$

where A and B are self-conjugate of degree k and $n-k$, respectively. Moreover

$$\mathrm{smr}_1(M') = \ldots = \mathrm{smr}_k(M') \neq \mathrm{smr}_{k+1}(M') = \ldots = \mathrm{smr}_n(M'). \tag{8.1}$$

Proof. Since the matrix M^t is distinguished, by Lemma 5.7 we get that the set $\{\mathrm{smr}_i(M) : 1 \leq i \leq n\}$ has at most two elements. Let $l = |\{i : \mathrm{smr}_i(M) = \mathrm{smr}_1(M)\}|$. If $2l \geq n$ take $k = l$ and $M'' = M$. Otherwise, construct M'' by conjugation of the first column of M – we get $\mathrm{smr}_1(M'') = \mathrm{smr}_1(M)$ and $\mathrm{smr}_i(M'') = \mathrm{smr}_i(M) + 1$ for $i > 1$. In both cases, letting $k = |\{i : \mathrm{smr}_i(M'') = \mathrm{smr}_1(M'')\}|$, we have $2k \geq n$.

There exists a permutation $\sigma \in S_n$, which fixes 1 and such that $M' = \sigma M''$ is of the block form

$$\begin{bmatrix} A & C \\ D & B \end{bmatrix},$$

where A, B are of degrees $k, n-k$ respectively and the equation (8.1) holds.

Let $U = \{i, j\}$ for $1 \leq i < j \leq n$. By our assumptions and Lemma 6.4 we have

$$1 = M'_{ii} + M'_{ij} + M'_{ji} + M'_{jj} + \mathrm{smr}_i(M') + \mathrm{smr}_j(M')$$

and hence

$$M'_{ij} + M'_{ji} = \mathrm{smr}_i(M') + \mathrm{smr}_j(M') + 1 \tag{8.2}$$

Consider two cases:

1. $j \leq k$ or $i > k$. This implies $i, j \leq k$ or $i, j > k$. Equation (8.2) gives us $M'_{ij} + M'_{ji} = 1$ and since M' is distinguished, $M'_{ij} = \overline{M'_{ji}}$. Hence A and B are self-conjugate.
2. $i \leq k < j$ and hence $\mathrm{smr}_i(M') = \mathrm{smr}_j(M') + 1$. Equation (8.2) gives us $M'_{ij} = M'_{ji}$, hence $D = C^t$. $\square$

Lemma 8.4. *Let $n \in \mathbb{N}$ and $M \in \mathcal{V}^{n\times n}$ be distinguished in the following block form*

$$M = \begin{bmatrix} A & C \\ C^t & B \end{bmatrix},$$

where A, B are of degrees k, l, respectively. Assume that $k > 0$ and:

1) A is self-conjugate;
2) $M_1 = [1, 2, \ldots, 2]$;
3) $J_M(\{1, i, j\}) \neq 0$ for $1 \leq i \leq k < j \leq n$.

Then C consists only of elements equal to 2.

Proof. If $l = 0$, there is nothing to prove. Assume that $l > 0$, take $i \le k$ and $j > k$. The principal submatrix of M defined by indices $1, i, j$ is of the form:

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & x \\ 2 & x & 1 \end{bmatrix}$$

If $x = 3$ then $J_M(\{1, i, j\}) = 0$, contrary to our assumptions, hence $M_{ij} = x = 2$. Together with the form of M_1, we get the desired result. □

Definition 8.5. Let $n \in \mathbb{N}$, $M \in \mathcal{V}^{n\times n}$ and S be a spin^c set for M. We will say that the spin^c pair (M, S) is in *standard form* if:

1) $S = \{1, \dots, |S|\}$;
2) $M_1 = [1, 2, \dots, 2]$;
3) M is distinguished and in the block form

$$\begin{bmatrix} A & 2 & * \\ 2 & B & * \\ * & * & * \end{bmatrix}$$

with elements on the diagonal of degrees k, l, r;
4) $k \ge l$ and $k + l = |S|$;
5) A, B are self-conjugate;
6) $\mathrm{smr}^S_1(M) = \dots = \mathrm{smr}^S_k(M) \ne \mathrm{smr}^S_{k+1}(M) = \dots = \mathrm{smr}^S_{k+l}(M)$ (it is possible that $l = 0$).

We can deduce some further restrictions on a standard form of a matrix.

Lemma 8.6. *Keeping the notation from the above definition, let (M, S) be a spin^c pair in the standard form and $k + l < m \le n$. Then, in the block form*

$$M_m = \begin{bmatrix} a & \bar{a} & * \end{bmatrix},$$

where $a \in \{2, 3\}$ is such that the equation

$$ka + l\bar{a} + (k - l - 1)2 = a \tag{8.3}$$

holds.

Proof. Let $i \le k + l$. Using the fact that (M, S) is a spin^c pair in the standard form and Lemma 6.4, for $U = \{i, m\}$ we get

$$\begin{aligned} 1 = \binom{|U|}{2} &= |(J_M(U) + U)S|_2 = |J_M(U)S|_2 + |US|_2 \\ &= [i \in S](M_{ii} + M_{mi}) + [m \in S](M_{im} + M_{mm}) \\ &\quad + \mathrm{smr}^S_i(M) + \mathrm{smr}^S_m(M) + [i \in S] + [m \in S] \\ &= M_{mi} + \mathrm{smr}^S_i(M) + \mathrm{smr}^S_m(M) \end{aligned}$$

and hence

$$M_{mi} = \mathrm{smr}^S_i(M) + \mathrm{smr}^S_m(M) + 1 = \begin{cases} \mathrm{smr}^S_1(M) + \mathrm{smr}^S_m(M) + 1 & \text{if } i \le k \\ \mathrm{smr}^S_1(M) + \mathrm{smr}^S_m(M) & \text{if } i > k \end{cases} \tag{8.4}$$

Set $a := M_{m1}$. Since M is distinguished, $a \in \{2, 3\}$ and the above equation gives us the desired form of the m-th row of M. This implies $\mathrm{smr}^S_m = ka + l\bar{a}$. In order to prove equation (8.3) notice, that $\mathrm{smr}^S_1 = 1 + (k + l - 1)2$ and use again formula (8.4) for $i = 1$. □

By the following lemma, certain spin^c pairs can be transformed to standard forms.

Lemma 8.7. *Let $n \ge 3$ be an odd integer and $M \in \mathcal{V}^{n\times n}$ be distinguished. Let S be a spin^c set for M. If*

$$J_M(U) \ne 0 \textit{ for } U \subset S \textit{ and } |U| = 3,$$

then there exists $g \in G_n$ such that (gM, gS) is a spin^c pair in a standard form.

Proof. By Corollary 8.2 (gM, gS) is a spin^c pair for any $g \in G_n$. Our goal is to show that (M, S) can be transformed to a pair in the standard form.

By permuting indices and conjugating columns, we can transform (M, S) to a form where $S = \{1, \ldots, |S|\}$ and $M_1 = [1, 2, \ldots, 2]$.

Let N be the principal submatrix of M defined on the set S. N is distinguished and for every $U \in \mathcal{P}(S) = \mathcal{P}_{|S|}$ we have

$$|J_N(U) + U|_2 = |(J_N(U) + U)S|_2 = |(J_M(U) + U)S|_2 = \binom{|U|}{2}.$$

In particular, $|J_N(U)|_2 = 1$ if $|U| = 2$. Using Lemma 8.3 we can act on M by an element of $G_{|S|} \subset G_n$ such that N becomes

$$N = \begin{bmatrix} A & C \\ C^t & B \end{bmatrix},$$

where A and B are self-conjugate of degrees k, l respectively, such that $k \geq l$ and

$$\mathrm{smr}_1(N) = \ldots = \mathrm{smr}_k(N) \neq \mathrm{smr}_{k+1}(N) = \ldots = \mathrm{smr}_{k+l}(N).$$

Note that $\mathrm{smr}_i(N) = \mathrm{smr}_i^S(M)$ for $1 \leq i \leq |S| = k + l$.

By assumption and Lemma 6.4 we have

$$J_N(U) = J_M(U)S = J_M(U) \neq 0$$

for $U \subset S$ and $|U| = 3$. By Lemma 8.4 we get that $C = 2$ and hence the spin^c pair (M, S) was transformed to a standard form. □

9. Spin^c structures on HW-manifolds

By the results of previous sections we know that the existence of a spin^c structure on an HW-manifold is equivalent to the existence of a spin^c set for its HW-matrix. We will show that this never happens in dimensions greater than 3.

Lemma 9.1. *Let $n \geq 5$ be an odd integer and $M \in \mathcal{H}_n$. There does not exist a spin^c set S for M such that $|S| = n$.*

Proof. If such a set S exists, then by our assumptions $J_M(U) \neq 0$ for $|U| = 3$ and by Lemma 8.7 we can assume that (M, S) is in a standard form:

$$M = \begin{bmatrix} A & 2 \\ 2 & B \end{bmatrix},$$

where the degrees of A, B equal k, l respectively, $k \geq l$ and:

$$\mathrm{smr}_i(M) = \begin{cases} 1 & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases}$$

By definition of HW-matrices we have

$$0 = \sum_{j=1}^{n} \mathrm{smc}_j(M) = \sum_{i=1}^{n} \mathrm{smr}_i(M) = k \cdot 1,$$

hence k is even and in particular $k < n$.

Let $U = \{1, \ldots, k\}$. Since it is of even size, $J_M(U)U = 0$ by Lemma 6.4. Moreover, for every $j > k$ we have

$$M_U[j] = \sum_{i \in U} M_{ij} = \sum_{i \in U} 2 = 0.$$

Hence $J_M(U) = 0$. Contradiction with the fact that $M \in \mathcal{H}_n$. □

Lemma 9.2. *Let $n \geq 5$ be an odd integer and $M \in \mathcal{H}_n$. There does not exist a spin^c set S for M such that $|S| = n - 1$.*

Proof. Similarly as in the proof of the previous lemma we can assume that

$$M = \begin{bmatrix} A & 2 & * \\ 2 & B & * \\ * & * & 1 \end{bmatrix}$$

where the matrices on the diagonal are of degrees $k, l, 1$ respectively, $k \geq l$ and $M_1 = [1, 2, \ldots, 2]$.

Since $k + l = n - 1$ is even, $k = l \mod 2$. By Lemma 8.6 we get

$$M_n = [a, \overline{a}, 1] \text{ and } k \cdot 1 + 2 = a.$$

If k is odd, then l is odd and $a = 3$. By definition of an HW-matrix, we get $\mathrm{smc}_i(B) = 1$ for some $k + 1 \leq i \leq k + l$ and

$$0 = \mathrm{smc}_{k+i}(M) = k \cdot 2 + \mathrm{smc}_i(B) + 2 = \mathrm{smc}_i(B) = 1,$$

a contradiction.

Assume that k is even. Then l is even and $a = 2$. If $l = 0$, then $M_n = [2, \ldots, 2, 1]$ and $J_M(\{1, n\}) = 0$, which cannot happen. Suppose $l > 0$. Take $U = \{1, \ldots, k\}$, $V = \{k + 1, \ldots, l\}$. They are both sets of even size. By the form of M and Lemma 5.7 we have

$$M_U[i] \in \{2, 3\} \text{ and } M_V[i] = l \cdot 2 = 0 \text{ if } i \leq k$$

and

$$M_U[i] = k \cdot 2 = 0 \text{ and } M_V[i] \in \{2, 3\} \text{ if } k < i < n.$$

Since M is an HW-matrix, we get $M_U[n] = M_V[n] = 1$, but then

$$0 = \mathrm{smc}_n(M) = M_U[n] + M_V[n] + 1 = 1,$$

a contradiction. □

Lemma 9.3. *Let $n \geq 5$ be an odd integer and $M \in \mathcal{H}_n$. There does not exist a spin^c set S for M such that $|S| = n - 2$.*

Proof. Similarly as in the previous two cases, we may assume that

$$M = \begin{bmatrix} A & 2 & * & * \\ 2 & B & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{bmatrix},$$

where the blocks on the diagonal are of degrees $k, l, 1, 1$, respectively and $k \geq l$. Since $k + l = n - 2$ is odd, $k = l + 1 \mod 2$. By Lemma 8.6 we have

$$M_{n-1} = [a, \overline{a}, 1, *] \text{ and } k \cdot 1 + \overline{a} = a,$$

hence $k \cdot 1 = 1$, k is odd and l is even.

Assume that $M_n = [b, \overline{b}, *, 1]$. We have $a \neq b$, otherwise

$$M_{n-1} + M_n = [0, \ldots, 0, c, d],$$

where $c, d \in \{2, 3\}$, hence $J_M(\{n - 1, n\}) = 0$.

For every $i \leq k$ we get

$$0 = \mathrm{smc}_i(M) = \mathrm{smc}_i(A) + l \cdot 2 + 2 + 3 = \mathrm{smc}_i(A) + 1,$$

hence $\mathrm{smc}_i(A) = 1$. But by Lemma 6.1 matrix A cannot exist, a contradiction. □

Proposition 9.4. *Let $n \geq 5$ be an odd integer and $M \in \mathcal{H}_n$. There does not exist a spin^c set for M.*

Proof. Let S be a spin^c set for M. By Lemmas 9.1, 9.2 and 9.3 we can assume that $|S| \leq n - 3$. In this case there exists a set $U \in \mathcal{P}_n$ of size 3 such that $US = 0$. By Lemma 6.4 $J_M(U) \subset U$, hence $(J_M(U) + U)S = 0$. Since S is a spin^c set for M, we have

$$0 = |(J_M(U) + U)S|_2 = \binom{|U|}{2} = \binom{3}{2} = 1,$$

a contradiction. □

Finally we are ready to state the main result of the paper:

Theorem 9.5. *Let X be a Hantzsche-Wendt manifold of dimension $n \geq 5$. Then X does not admit a spin^c-structure.*

Proof. This follows directly from Theorem 7.6 and Proposition 9.4. □

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